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The Geometric Index of the Wheel, W_n

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Abstract

The geometric index of a graph G is defined as the smallest non-negative integer n such that a graph G is a unit graph in r. Graphs considered are finite, undirected, without loops nor multiple edges. Also, edge crossings are allowed in the figures but distinct vertices must have distinct coordinates and that the line segment joining adjacent vertices must not pass through any other vertex. In this paper, the geometric index g of the wheel W_n (n 3, 4, 5,…) is discovered and proven. The results of this study may serve as a benchmark information to other researchers interested in expanding the study of geometric index on all graphs.

Keywords: unit graphs, unit embedding

Introduction

A unit graph in Rn is a graph which can be drawn on the n-space such that the length (Euclidean distance between the end-vertices of the edge) of every edge (if there is any) is equal to one unit. The problem of finding the geometric index of a graph G is quite new. It was first thought of by Gervacio (1996), and thus far no earlier study has been made in this problem. Hence, the results of this research, which is a pioneering study, may provide future researchers a valuable guide to the solution of the general problem of finding the geometric index of all graphs. This study considers only graphs which are finite, undirected, without loops nor multiple edges. Edge crossings are allowed in the figures but distinct vertices must have distinct coordinates and that the line segment joining adjacent vertices must not pass through any other vertex. This study aims to find the value of n for which some special graphs can be called a unit graph in r.

Discussion

Some preliminary concepts ofthis study follow.

Definition 1: A *graph* (Fig. l) is a pair (V(G), E(G)) where V(G) is a nonempty finite set and E(G) is a set of unordered pairs of distinct elements of V(G). The elements of V(G) are called the *vertices* and the elements of E(G) are *edges* (Harrary, 1969).

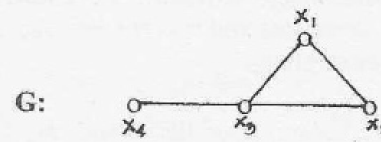

Figure 1 : G is a graph where $V(G) = \{x_1, x_2, x_3, x_4\}$ and $E(G) = \{[x_1,x_2], [x_2, x_3], [x_1, x_3], [x_3, x_4]\}$.

Definition 2: A graph G is said to be a unit graph in R^n if all the edges of G are of length I or if every two adjacent vertices x_i, x_j of G, d $(X_1, X_1,) = 1$.

Definition 3: The geometric index of a graph G, denoted by g(G), is the smallest integer n such that G is a unit graph in r.

A special graph can now be examined.

Definition 4: The wheel W_n of order n + 1 is the sum of the cycle, C_n $[V_1, V_{2...,} Vn, V_1]$ and K_1.

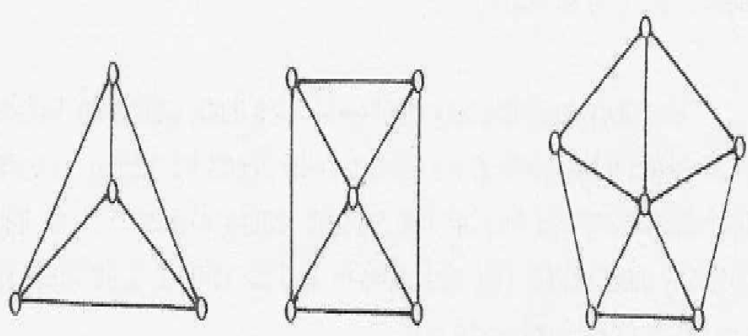

Figure 2: Some wheels of orders $4(W)_3$, $5(W_4)$ and $6(W_{5)})$

Theorem 6: The wheels $W_{3,}$ W_4 and W_5 are unit graphs in R^3 and $\mu(W_3) = \mu(W_4) = \mu(W_5) = 3$

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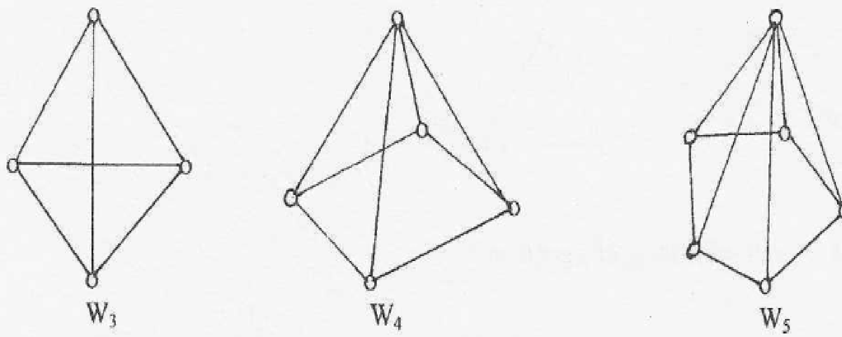


Figure 3: The wheels W_3, W_4 and W_5 drawn in R^3

Observe that the wheels W_3, W_4 and W_5, are not unit graphs in R2 but the wheel W6 is a unit graph in that same space as formally stated in Theorem 7.

Theorem 7: The wheel W_6 is a unit graph in R^2 and $\mu(W_6) = 2$.

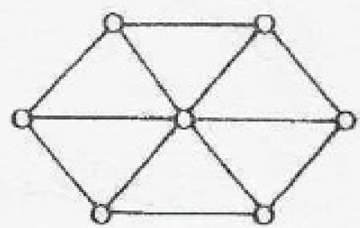


Figure 4: The wheel W_6 drawn in R^2

The following concepts prove that for any wheel Wn ($n \geq 7$), its geometric index is 3.

Definition 8: A unit *embedding of graph G* in the Euclidean space R^n is a one-to-one mapping Φ: $V(G) \rightarrow R^n$ such that $d(\Phi(x), \Phi(y)) = 1$ for all $[x, y]$ $\varepsilon E(G)$

Remark 9: It is useful to think of a unit embedding of a graph G in R^n that would result in a unit graph in R^n.

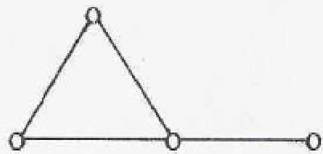
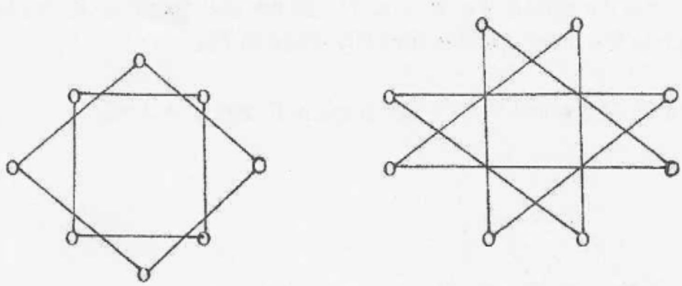


Figure 5. A unit embedding of a graph in R^2

Definition 10: Let $n \geq 3$ and let $k < n$ be any integer not divisible by n. We define the graph G (n, k) to be the graph with vertices 1, 2,…,n whose edges are [*i, i* + *k*], where k = The sum *i* + *k* is to be read modulo n.



G(8,2) G(8,3)

Figure 6: Some graphs G(n,k)

Lemma 11: Let $n \geq 3$ and let k be an integer not divisible by n. Then G(n,k) is a cycle if and only if n and k are relatively prime.

Proof:

First, assume that n and k are relatively prime. We claim that the sequence [1, 1+k, 1 +2k, 1+(n-l) k] is a path in G (n, k)- The elements of the sequence are to be read modulo n. It is clear that consecutive vertices in the sequence are edges of G, by definition. Suppose that two vertices in the path are equal, say 1 +*i*k ≡ 1 + *j*k (modn). Then n/ [(1+*i*k)- (1-*j*k)]. Then n / (*i*k-*j*k) implying that n / k(*i*-j). Since n and k are relatively prime, n / (*i* -*j*) and by definition *i*≡*j(*mod n). But 0 ≤ *I* ≤ n-1 and 0 ≤ *j*≤ *n*-1. It follows that *i*≡*j*. Now, the last vertex of the path is adjacent to the first vertex 1 because I + (n- l) k + k = 1 + nk I(mod n). Therefore, G(n,k) is a cycle.

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Next, let us assume that n and k are not relatively prime and let their greatest common divisor be equal to d > 1. Let n' = $\frac{n}{d}$ and consider the sequence [l, 1+ k, I + 2K,…, 1+(n'-1)k]. Consecutive vertices of this sequence are adjacent by definition of G. If 1 + *i*k = 1 +*j*k(mod n), we would have 1k ≡ *j*k (mod n). If we divide the congruence k(*i-j*) through by d, n / (tk —Jk) implying that $\frac{nc}{d} = \frac{k(i-j)}{d}$ for some c ε Z, then we have $\frac{nc}{d} = \frac{k}{d}(i-j)$. Since $(\frac{n}{d}, \frac{k}{d})$= 1 then $\frac{n}{d}/(i-j)$. But $\frac{n}{d} = n'$. Therefore, i≡j (mod n'). But 0 ≤ n'-1 and 0 ≤ *j* only. It follows that i = j. The last vertex of the path is adjacent to the first vertex 1 since 1 + (n'-1) k+k = 1 + n'k = 1+ (n/d) k = 1 + n(k/d) 1(mod n). Therefore G (n, k) contains as a subgraph a cycle of length n' which is less than n. Consequently, G (n, k) is not a cycle.

Hence, G is a cycle if and only if n and k are relatively prime.

Lemma 12: The wheel W_n (n ≥ 7) is not a unit graph in R^2.

Proof:

By Lemma 11, we know that G(n,k) is a cycle if and only if (n,k) = 1. Consider the cycle G(n,k) and let P_1, P_2, pn be n points in the plane forming consecutive vertices of a regular n-gon. Furthermore, let P_1 be associated with vertex i of G(n, k).

<u>Case l:</u>

Suppose that the n-gon is of such size that the distance between P_1 and be equal to 1 unit. Clearly, G(n,k) is a unit graph in R^2. The center of the n-gon forms an isosceles triangle with and P_{1+k}. Furthermore, the angle of this triangle at the center of the n-gon is $\frac{360k}{n}$ degrees and since n ≥7, it follows that the distance from the center to n is less than 1 unit. Thus, W_n is not a unit graph in R^2.

Case2:

Suppose the n-gon is constructed in a unit circle such that the distance between the center and to any other vertex of the n-gon is 1 unit. Using the same argument as above, it follows that the distance between 2 adjacent vertices of the n-gon is greater than 1 unit. Thus, W_n is not a unit graph in R^2.

Hence, in either case W_n is not a unit graph in R^2

Theorem 13: For each n ≥ 7, μ(W_n) = 3.

Proof:

For n ≥ 7, let n = 2^rm where r ≥ 0 and where m is odd. Consider the following cases:

Case 1: r = 0.

Note that m and m-1 are relatively prime since these are consecutive integers. Consequently, m and k = $\frac{m-1}{k}$ are also relatively prime. By Lemma 11, G(n,k) is a cycle. Let P_1, P_2 be n points in the plane forming consecutive vertices of a regular n-gon. Furthermore, let P_i be associated with vertex i of G(n,k). We assume that the n-gon is of such size that the distance between P_1 and P_{1+k} be equal to one unit. We therefore have a unit embedding of the cycle G(n,k) in R^2. The center of the n-gon forms an isosceles triangle with P_1 and $P_{1+k.}$ Furthermore, the angle of this triangle at the $\frac{360k}{n} = \frac{180(n-1)}{n} = 180 - \frac{180}{n}$ degrees and since n > 6, we have $150 < 180 - \frac{180}{n} < 180$ degrees. It follows that the distance from the center to p_1 is less than 1 unit. We can therefore locate p_o above the center of the n-gon such that p_o is at a distance 1 from each of the vertices of the n-gon. Therefore, we have a unit embedding of W_n in R^3.

Case 2: r > 1.

Take $k=2^{r-1}m - 1$ so that k and $n = 2^r m$ are relatively prime. Consider the cycle G(n,k) and associate the vertices 1,2,…,n of this cycle with points $p_1, p_2, \ldots, p_n$ in the plane forming the vertices of a regular n-gon. Let the size of the n-gon be such that the distance between P_1 and P_{1+k} equal to 1 unit. Then, we have a unit embedding of the cycle G(n,k) in R^2. The center of the n-gon forms an isosceles triangle with P_1 and P_{1+k}. The angle at the center is equal to $\frac{360k}{n}$ degrees and $120 < \frac{360K}{n} < 180.$ The rest of the argument is similar to that in Case 1.

Case 3: r = l.

Take k = m - 2 so that k and n = 2m are relatively prime. Consider the cycle G(n,k) and associate the vertices of this cycle with points $p_1, p_2, \ldots p_n$ in the plane forming the vertices of the regular n-gon. Let the size of the n-gon be such that the distance between p_1 and p_{1+k} be equal to one unit. Then, we have a unit embedding of the cycle G(n,k) in R^2. The center of the n-gon forms an isosceles triangle with p_1 and $p_{1+k.}$ The angle at the center is equal $\frac{360k}{n}$ degrees and $100 < \frac{360k}{n} < 180$. It follows that the distance from the center p_0 to p_1 is less than 1 unit. We can therefore locate a point p_o above the center of the n-gon such that p_o is at a distance 1 from each of the vertices of the n-gon.

Hence, in any case, we can find a unit embedding of W_n in R^3. Since there is no unit embedding of W_n in R^2 from Lemma 12, it follows that $\mu(W_n) = 3$.

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Summary, Conclusions and Recommendations

When n=3,4, or 5, the geometric index of the wheel W_π is 3. When n=6, $\mu(W_6) = 2$ and for $n \geq 27$, the smallest dimensional space where Wn can be drawn as unit graph is in R^3.

The geometric indices of some other graphs were also discovered like the path, the ladder, cycle which the readers may examine on their own.

Other areas of geometric index of graphs includes other areas which the researcher has not examined. Hence, for mathematicians who are interested in graphs, the following questions are recommended for further studies:

1. What are the geometric indices of graphs like the double fan $P_n + \overline{K2}$, the torroidal grid C_m x C_n, the mobius ladder M_n and the double cone $C_6 + \overline{K2}$?

2. How can we construct the petersen graph in R^3 so that $\mu(Ps) = 3$?

3. How can we characterize graphs which can be drawn in R, R^2, R^3,..., R^n ?

4. Can we find a formula in obtaining the geometric index of a graph G using some graph invariants?

Literature Cited

Baylon, Rowena T. 1996. On the Geometric Index of Graphs. Unpublished M.S. Thesis. MSU-IIT, Iligan City.

Gervacio, S.V. 1995. Unit Graph in Euclidean n-Space. Professional Chair Lecture, De La Salle University, Manila. 5 pp.

Gervacio, S.V. 1995. New Outlook in Mathematics: Discrete vs. Continuous. Plenary Paper. In: Mathematics Teacher Education Conference, Ateneo de Manila University, Manila.

Harrary, F. 1969.Graph Theory. Addison-Wesley Publishing Co. Massachusetts.506 pp.