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Denumerability of the Algebraic Numbers

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Abstract

An algebraic number is a real number that is a root of a polynomial equation $\alpha_n X^n + \alpha_{n-1} X^{n-1} \ldots + \alpha_0$ where α_i are integers. In this paper, using the fact that a polynomial equation of degree n has at most n roots, together with some results, the denumerability of the algebraic numbers is proven.

Keywords: denumerability, equinumerous, algebraic number

Introduction

A set A is countable (or denumerable) if it is finite or equinumerous with the set of natural numbers, otherwise it is uncountable. In the number system, some sets which are subsets of the set of real numbers are finite and some are infinite. Further, these sets may be either denumerable or nondenumerable. Some results as to the denumerability of sets are given by Lipschutz (1981). In addition, the discussions presented here can be used to prove the denumerability of the algebraic numbers. This problem is outlined in the book of Pownall (1994). It is in this paper that the researcher aimed to prove the claim that the set of algebraic numbers is denumerable.

Discussion

Definition 1: A set is countable (denumerable) if it is finite or equinumerous with the set of natural numbers, otherwise it is uncountable,

Definition 2: Two sets A and B are equinumerous or equal in cardinality if there exists a one-to-one correspondence between their elements.

Definition 3: A set A is finite if there exist some natural numbers n such that $A =_c \{i/i<n\}$. Otherwise, A is infinite.

Definition 4:

b.) A countable $\Longleftrightarrow A =_c N$
c.) f: $N \rightarrow A$ and f is onto $\Leftrightarrow$ A is countable.
d.) g: $A \rightarrow B$ and g is one-to-one and B is countable $\Rightarrow$ A is countable.

Definition 5: A real number is algebraic if it is a root of some polynomial(x) = a0 + al X + . . . + an Xn where ai € n ≥1, an≠0.

Lemma 6: N X N is countable.

Proof:

To exhibit an injection from N x N to N:
Define g: N x N ⇒ N such that g (n, m) = p^n . qm where p and q are fixed primes.

Suppose g [(n, m)] = g [(r, s)] ⇒ p^n. q^m = p^r. q^s.
By the uniqueness of the canonical prime factorization of integers, n = r and m = s ⇒ (n, m) = (r, s). Therefore, g is 1-1 function.

Hence, N x N is countable.

Definition 7: Let A and B be sets, then the Cartesian Product of A and B is defined as follows:

A XB = {a, b}: a€A {b€ B}

Lemma 8: If A and B are' denumerable sets, then A X B is also denumerable:
Proof:
Since and B are denumerable, we can list their elements,
A = {a_1, a_2, …, a_n, …}, B = {b_l, b_2, …, b_n, …}, in such a way that if m ≠ n, then am ≠ an, and bm ≠ bn. The elements of A x B are the ordered pairs (ai, bi) where i, j € z^+.

We arrange these ordered pairs in the following array:

(a_i, b_i) (a_i, b_2) (a_i, b_3) (a_i, b_4) …
(a_2, b_i) (a_2, b_2) (a_2, b:) (al, b,) …
(a3, bi) (a3, b2) (ag, b3) (a3, b4) …
(a4, bi) (a4, b2) (a4, b3) (a4, b4,) …

It follow-s from the equality property for ordered pairs that no two of these ordered pairs are equal, it is sufficient then to devise a way of listing them.
A simple procedure is
AXB= {C_1, C_2, C_3, …, C_n…}

Where c1 (a1bl), C2= (a2, b1), C3= (al, b2), C4 = (a3, bl), C5 = (a2.b2),
C6 = (al, b3), C7 = (a4, b1), C8= (a3, b2), c9= (a2, b3),
C10= (al, b4).

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In Figure 1, we start at the upper left comer of the array, then follow along the upward slanting lines as indicated by the arrows. Each ordered pair is assigned a unique positive integer, and each positive integer, and each positive integer determines a unique ordered pair. Thus, the lemma is proved.

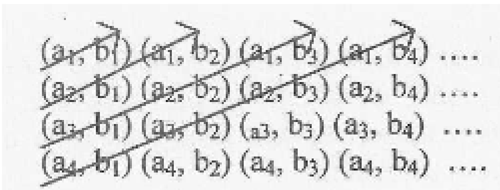

Figure 1. Listing A x B

Lemma 9: the set of polynomials of degree n (fixed) with integer coefficients is denumerable.

Proof:

For each pair of positive integers (n, m) ∈ N x N, let Pnm denote the set of polynomials p(x) of degree m in which

$| a_o | + | a_1 | + \ldots + | a_m | = n$ observe that P_{nm} is finite. Accordingly, P = U $\{P_{mn}$: (n, m) ∈ N x N$\}$ is countable since it is a countable union of countable sets. In particular, since P is not finite, P is denumerable.

Lemma 10: The set of roots of polynomial equations of degree n (fixed) with integer coefficients is denumerable.

Proof:

Note by Lemma 9 that the set F, of polynomials is denumerable.

$E = \{p_1(x)=0, p_2(x) = 0, p_3 (x) = 0, \ldots\}$

Let A_i = {x. x is a solution of Pi (x) = 0}. Since a polynomial of degree n can have at most n roots, each A, is finite. Hence, A = U {A*i*. *i* ∈ IN} is denumerable.

Theorem II: the set of algebraic numbers is' denumerable.

Proof:

Let ∏ = {P (x) | p is a polynomial over Z}.
Then ∏ = c Z^n via
a0+a1+a2x3 ⇒ (αo, αl. α2, …, an). Thus:
Thus: ∏ is countable.

Each polynomial in 11 has at most n roots over the set of real numbers (R). Thus, we only have a finite number of algebraic numbers that each polynomial in I-I will contribute. Thus, a countable set. Take the union of all these countable sets, then we obtain a countable set.

Conclusion and Recommendation

With the definitions and the proofs of lemmas, the denumerability of algebraic numbers is proved.

It is recommended for further studies to explore the proof of the uncountability of the set of transcendental numbers.

Literature Cited

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