Volume 5 Issue No. 1 December 2002
P-ISSN: 2244-4432 • E-ISSN: 2984-7125
National Peer Reviewed Journal
DOI: https://doi.org/10.59120/drj.v5i1.56

Topology and Metricity of the Complex Plane

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Abstract

The equinumerosity of the set of complex numbers and a set of ordered pairs of real numbers is used to show the topology of the complex plane. Likewise, the metricity of the complex plane is proved with the aid of the metricity of the plane R.

Keywords: topology, metricity, complex numbers, equinumerosity

Introduction

Many concepts in topology are abstraction of properties of sets of real numbers. The geometric representation of the set of real numbers, R is a set of points on a straight line. Similarly, R^2 is a plane consisting of a set of infinitely many points and each point denoted by an ordered pair (x, y) of real numbers.

Every complex number z= x + {y can be represented geometrically as a point on the xy plane, that is, the complex plane. The equinumerosity of the set of complex numbers and a set of ordered pairs of real numbers, denoted by R^2 will lead us to prove the topology of complex plane. However, the distance formula of complex number and metricity of the plane R^2 is used to prove the metricity of the complex plane.

Discussion

To present and discuss the topology of the, complex plane, Theorem 1 and the concept of topology of the plane R^2 are used.

Theorem 1. A set C is equinumerous to a set R^2, if there exists a one-to-one correspondence between their elements.

Open Disc and Distance Formula

First, define a set in C is defined. If Z_o is a point in the plane and ε is a positive number, the disc $\{z: |z-z_o| < \varepsilon\}$ is called the open disc. An open disc D_z in the complex plane is the set of points inside a circle, with center $z_o = (x_o, y_o)$ and radius e > O. That is, $\{(x, y): (x - x_o)^2 + (y- y_o)^2 < \varepsilon^2\}$ $\{z \ \varepsilon \subset: d(z_o, z) < \varepsilon\}$ (Greenleaf, 1972; Fig. 1).

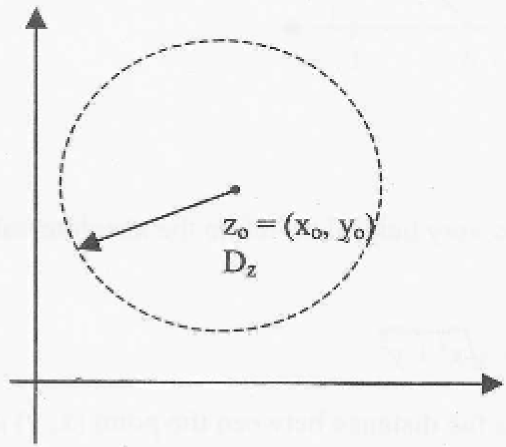

Figure 1. Open disc

The open disc plays a vital role in the topology of the plane R^2 as well as in complex plane, that is, analogous to the role of the open interval in the topology of the R.

The distance between two points (x_o, y_o) and z = (x, y) in C is denoted by d (zo, z). The distance formula for a complex number IS derived by presenting and discussing its geometric interpretation.

Complex number Z can be defined as ordered pairs z = (x, y) of real numbers x and y. Each complex number corresponds to just one point in a complex plane of the z plane (Fig. 2). The complex number z can also be thought as the directed line segment, or vector from the origin to any point (x, y).

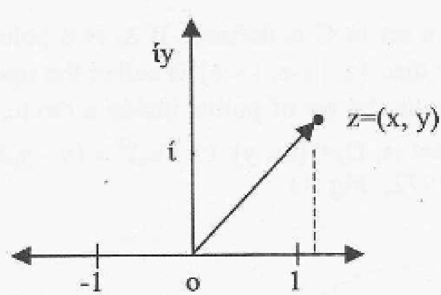

Figure 2. Point z and vector

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In discussing z | of the. complex nur

$$|z| = |x + iy| = \sqrt{|x|^2 + |y|^2} = \sqrt{x^2 + y^2}$$

Geometrically the number | z | is the distance between the point (x, y) and the origin, or the length of the vector representing z.

Proceeding to the goal which is to derive the distance formula for complex number, denote the distance from a point $z_o = x_o + y_o$ to another point $z_o = x_o + iy_o$ in C, that is, $| z_o - z |$ is the length of the vector representing z_o-z.

Then, to derive the distance formula for C, alternatively follow the expression:

$$z_0 - z = (x_0 + iy_0) - (x + iy)$$

$$\Rightarrow z_0 - z = x_0 + iy_0 - x - iy$$

$$\Rightarrow z_0 - z = (x_0 - x) + i\,(y_0 - y)$$

Since the goal is the distance which can be considered as the absolute value, then;

z_0-z can be,

$$|z_0 - z| = |(x_0 + iy_0) - (x + iy)|$$

$$\Rightarrow |z_0 - z| = |x_0 + iy_0 - x - iy|$$

$$\Rightarrow |z_0 - z| = |(x_0 - x) + i\,(y_0 - y)|$$

$$\Rightarrow |z_0 - z| = |x_0 - x|^2 + |y_0 - y|^2$$

$\Rightarrow |z_0 - z| = \sqrt{(x_0 - x)^2 + (y - y)^2}$ or just equal to

$$d(z_0, z) = \sqrt{(x_0 - x)^2 + (y_0 - y)}^2$$

Finally, the distance formula for complex numbers, is derived as:
$|z_0 - z| = \sqrt{(x_0 - x)^2 + (y_0 - y)^2}$ or just equal to $d(z_0, z) = \sqrt{(x_0 - x)^2 + (y_0 - y)^2}$

Interior Point

Our basic tool is the concept of an ε neighborhood-of a given point z_o. It contains all points z lying inside but not on a circle centered at z_o and with a specified radius ε (Fig. 3).

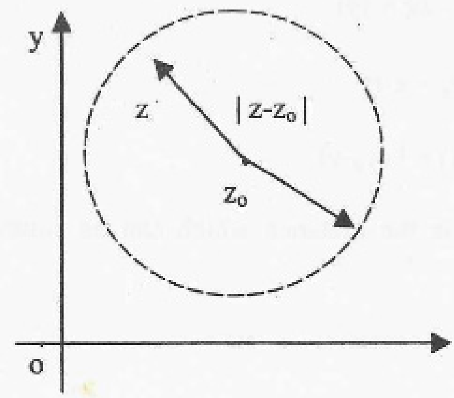

Figure 3. Interior point

A point z_o is said to be an interior point of a set S whenever there is some neighborhood of zo that contains only points of S (Churchill and Brown, 1984). The-interior of a set S, is the set of all points z_o such that S is a neighborhood of z_o. We may also consider an open disc, (i.e., let $S \subset C$). A point z_o is an interior point of S if and only if zo belongs to some open disc D which is contained in S.

i.e.: $z_o \, \varepsilon \, D_z \subset S$

$S \subset C$ has interior points that means $S \neq \emptyset$.

Exterior Point

A point z_o is called as exterior point of S when there exists a neighborhood of it containing no points of S. A point zo is an exterior point of S if and only if z_o does not belong to some open disc D_z which is contained in S.

i.e., $Zo \notin D_z \subset S$

Every point outside the open disc $\{z: | z - z_o | < \varepsilon\}$ in the complex plane is the exterior point of S.

Boundary Point

A boundary point is a point all of whose neighborhoods contain points in S and points not in S. If z_o is neither an interior nor an exterior point, then it is the boundary point of S, that is, the circle | z | = r is the boundary of each of the sets | z | < r and | z | r in the complex plane. But the set of all complex numbers has no boundary point.

Open Set

A subset S in C is called an open set if it has the following property. For each point z_o in S there is some open disc $\{z: | z - z_o | < \varepsilon\}$ with positive radius ε about z_o which lies entirely within S. The radius will depend on which point zo we are looking at (Greenleaf, 1972).

A set is open if it contains none of its boundary point. For a set to be open, there must be a boundary point, that is, not. contained in a set.

The set is open if and only if each of its points is an interior point. Let S be open, thus every interior points z_o, $z_o \varepsilon S \subset S$. Conversely, suppose interior S=S since interior of S is open it follows that S is open (Wilansky, 1970).

Closed Set/Closure of a Set

A set is closed if it contains all its boundary points. For instance, $| z | \leqq r$ the closure of a set $S \subset C$ is the closed set consisting of all points in S together with the boundary of S.

For a set to be closed there must be a boundary point that is contained in the set. Hence the closed disc in the complex plane is also a closed act.

A subset S of C is closed if and only if its complement S is an open subset of C.

Connectedness and Disconnectedness/Domain

An open set S is connected if each pair of points z_1 and z_2 in it can be joined by a polygonal path consisting of a finite number of line segments end to end, which lies entirely in S (Churchili and Brown, 1984).

The open set | z | < r is connected. An open set that is connected is called a domain. Note that any neighborhood is a domain. A domain together with some, none, or all of its boundary points is referred to as a region.

In this manner, we can say that an open disc $\{z: | z - z_o | < \varepsilon\}$ or the whole complex plane are connected, and a connected open set in the plane is called a domain.

Boundedness of a Complex Plane

A set is bounded if every point of S lies inside some circle | z | = r, otherwise it is unbounded. A subset S in the complex plane is bounded if it lies within some disc of finite radius about z=0, that is, the absolute value $|z|$ is bounded as z varies through points in S, otherwise we say that S is an unbounded set (Greenleaf, 1972).

Thus, $|z| \leq r$, are bounded sets, while $|z| > R$ and the set of C, are unbounded.

Accumulation Point

A point z_o is said to be an accumulation point of set S, if each neighborhood of z_o contains at least one point of S distinct from z_o. It follows that if a set S is closed, then it contains each of its accumulation points.

A point $z_o \in C$ is an accumulation point or limit point of a subset S of C if and only if every open set A containing z_o contains a point of S different from z_o.

i.e.: $A \subset C$ open, $z_{o\,\in}\ S \Rightarrow S \cap (A \setminus \{z_o\}) \neq \emptyset$

A point zo is not an accumulation point of a set S whenever there exists some neighborhood of z_o which does not contain points of S distinct from z_o.

Some theorems regarding the topology of the complex plane are now presented.

Theorem 2. The union of any number of open subsets of C is open.

Proof: Let I be some set, and suppose given for each $i \in i$ and an open set S_i. Let S be the union of the S_i that is the set of all z_o such that to each $z_o \in S_i$, for some I, then S is open. Since given $z_o \in S$, we know that $z_o \in S_i$ for some I, so there exist an open disc D_z centered at z_o such that $z_o \in D_z \subset S_i \subset S$, and therefore S is open.

Theorem 3. The intersection of any finite number of open subsets of C is open.

Proof: Let S be a finite collection of open sets, say $S = \{S_1, S_2, \ldots .S_n\}$. To say that z_o belongs to the intersection means that z_o belongs to each S_1 ($i = {}_{1,2,\ldots}n$). since all of the sets S_1 are open, there exists for each $i = {}_{1,2,\ldots}n$ a positive number a: such that, the collection of sets whose members are neighborhoods of z_o is a subject of S_1 $\{N\alpha:(Z_0) \leq S_i\}\alpha_1, \alpha_1 \ldots, \alpha$ n, then $\alpha>$ and $N\alpha\,(z_o)$ is contained in every $N\alpha i(z_o)$; thus $N\alpha$: (Z_o) $S_1, S_2, \ldots S_n$.

Corollary 4. The intersection of two open subsets of C is open.

Proof: Given $z_c \in S_1 \quad S_2$, there exists an open disc D_z of radius r centered at zo contained in S_1, and there exist an open disc $D_z^{\,l}$ of r centered at z_o contained in S_2. Let a = min (r, r^l). Then the open disc of radius a centered at z_o contained in $S_1 \quad S_2$, which is therefore open.

Theorem 5. The subsets S of C is closed if and only if S contains each of its

accumulation points.

Proof: Assume that S of C is closed. If z_o is an accumulation point, then any open disc centered at zo cannot be in the complement of S. Hence zo lies in S.

Conversely, assume that S contains all of its accumulation points. Let x_o be in the complement of S. Then x_o is not accumulation point of S, and so there exist some open disc centered at x_o whose intersection with S is empty. Hence the complement of S is open, thereby proving the theorem.

Some concepts, properties and theorems regarding the topology of complex plane has been presented and discussed. We have proven therefore the topology of the complex plane.

Metricity of the Complex Plane

The complex plane is a metric is shown next, by satisfying the following definition and axioms.

Let S be a non-empty subset of C. A real valued function d defined on C that is elements in S, is called a metric or distance function on S if and only if it satisfies the following axioms:

M_1 : $d(z_1, z_2) \geq O$ and $d(z_1, z_1) = O$

M_2 : (symmetry) $d(z_1, z_2) = d(z_2, z_1)$

M_3 : (Triangle Inequality) $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$

M_4 : If $z_1 \neq z_2$ then $d(z_1, z_2) > 0$

The complex number $d(z_1, z_2)$ is called distance from z_1 to z_2.

Observe that a complex number $z = x + iy$ can be represented as a point in the plane R^2 and this complex number corresponds to ordered pairs (x.y) of R^2. In this way, we denote $Z_1= (X_1, Y_1)$ and $z_2=(X_2,Y_2)$ i.e., points of the complex plane.

To prove that complex plane is a metric, the following axioms must be satisfied;

M_1 : $d(z_1, z_2) \geq 0$ and $d(z_1, z_1) = 0$

Proof: Let $z_1 = x_1 + iy_1 \Rightarrow (x_1, y_1)$ and $z_1 = x_2 + iy \Rightarrow (x_2, y_2) \in C$. First, show that $d(z_1, z_2) \geq o$. Suppose that z_1 and z_2 are at different points. Using the distance formula for complex number, $d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \geq 0$. The distance between two points is never negative, that there correspond a unique positive complex number. Thus $d(z_1, z_2) \geq 0$. Also show that $d(z_1, z_1) = 0$. Observe that $d(z_1, z_1) = \sqrt{(x_1 . x_1)^2 + (y_1 - y_1)^2} = \sqrt{(0)^2 + (0)^2} = 0$. The distance between z_4 with respect to itself is equal to zero. Thus $d(z_1, z_2) = 0$. Hence $d(z_1, z_2) \geq 0$ and $d(z_1, z_1) = 0$.

M_2 : (symmetry) $d(z_1, z_2) = d(z_2, z_1)$

Proof: Let $z_1 = x_1 + iy_1 \Rightarrow (x_1, y_1)$ and $z_1 = x_2 + iy_2 \Rightarrow (x_2, y_2) \in C$. Note that the distance between z_1 and z_2 in the complex plane is denoted by $d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Show that $d(z_1, z_2) = d(z_2, z_1)$.

Observe that,

$$
\begin{aligned}
d(z_1, z_2) &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{[-(x_2 - x_1]^2 + [-(y_2 - y_1)]^2} \\
&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= d(z_2, z_1)
\end{aligned}
$$

Thus the distance from a point z_1 to a point z_2 is the same as the distance from z_2 to z_1. i.e.; $d(z_1, z_2) = d(z_2, z_1)$.

M_3 : (Triangle Inequality) $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$

Proof : Let $z_1 = x_1 + iy_1 \Rightarrow (x_1, y_1)$ and $z_2 = x_2 + iy_2 \Rightarrow (x_2, y_2) \in C$. The distance between point z_1 and z_2 in the complex plane is denoted by:

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (v_1 - y_2)^2}$$

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Define another point $z_3 = z_3 + iy_3 \Rightarrow (x_3, y_3) \in C$.

The distance between point z_2 and z_3 in complex plane is denoted by;

$$d(z_2,z_3) = \sqrt{(x_2-x_3)^2 + (y_2-y_3)^2};$$

To show that $d(z_1, z_3) \leq d(z_1, z_3) + d(z_2, z_3)$, define $d(z_1,z_3)=|z_1-z_3|$.

For any $z_1, z_2, z_3 \in C$. Now $d(z_1, z_3) = |z_1 - z_3|$

$$= |z_1 - z_2 + z_2 - z_3|$$

$$= |(z_1 - z_2) + (z_2 - z_3)|$$

$$\leq |z_1 - z_2| + |z_2 - z_3| \text{ (Triangle inequality)}$$

$$= d(z_1, z_2) + d(z_2, z_3)$$

Thus the length of $d(z_1,z_3) \leq d(z_1, z_2) + d(z_2, z_3)$.

M_4 : If $z_1 \neq z_2$ then $d(z_1,z_2) > 0$.

Proof : Let $z_1 = x_1 + iy_1 \Rightarrow (x_1,y_1)$ and $z_2=x_2+iy_2 \Rightarrow (x_2,y_2) \in C$

Suppose $z_1 \neq z_2$ then from distance formula of complex number $d(z_1,z_2)=$

$\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} > 0$. Moreover the distance between two distinct points in complex plane is greater than zero. Therefore $d(z_1,z_2)>0$.

Since the given four axioms of metricity are satisfied, hence the complex plane is a metric.

Summary

With the aid of the result of equinumerosity of C and R^2 and some concepts regarding the topology of R and R^2 and metricity of plane R^2, the topology of the complex plane is proved. Some properties and theorems on the topology of R and R^2 are presented and discussed. Furthermore, we have done the same to the topology of the complex plane by supplementing some theorems regarding the set of complex numbers.

The distance formula for the set of complex numbers was derived. The complex plane is proved to be a metric by satisfying its axioms. Thus, the topology and metricity of the complex plane are proved. Based on the results of this study, it is recommended that further study be made on the topology of R^{Π}.

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